About linear superpositions of special exact solutions of Veselov-Novikov equation

V. G. Dubrovsky[1], **A. V. Topovsky**[2]

Novosibirsk State Technical University, Karl Marx prosp. 20, Novosibirsk 630092, Russia

E-mail: [1]dubrovsky@ngs.ru

E-mail: [2]topovsky.av@gmail.com

Abstract. New exact solutions, nonstationary and stationary, of Veselov-Novikov (VN) equation in the forms of linear superpositions of arbitrary number of exact special solutions $u^{(n)}$, $n = 1, \ldots, N$ are constructed via $\overline{\partial}$-dressing method in such a way that the sums $u = u^{(k_1)} + \ldots + u^{(k_m)}$, $1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant N$ of arbitrary subsets of these solutions are also exact solutions of VN equation. The presented linear superpositions include as superpositions of special line solitons with zero asymptotic values at infinity and also superpositions of special plane wave type singular periodic solutions. By construction these exact solutions represent also new exact transparent potentials of 2D stationary Schrödinger equation and can serve as model potentials for electrons in planar structures of modern electronics.

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1. Introduction

Among different (2+1)-dimensional integrable nonlinear equations [1–9] prominent place takes the famous Veselov-Novikov (VN) equation [10, 11]:

$$u_t + \kappa u_{zzz} + \overline{\kappa} u_{\bar{z}\bar{z}\bar{z}} + 3\kappa(u\partial_{\bar{z}}^{-1}u_z)_z + 3\bar{\kappa}(u\partial_z^{-1}u_{\bar{z}})_{\bar{z}} = 0 \tag{1.1}$$

where $u(z, \bar{z}, t)$ is scalar function, κ is some complex constant; $z = x + iy$, $\bar{z} = x - iy$; ∂_z^{-1} and $\partial_{\bar{z}}^{-1}$ are operators inverse to ∂_z and $\partial_{\bar{z}}$, $\partial_{\bar{z}}^{-1}\partial_{\bar{z}} = \partial_z^{-1}\partial_z = 1$.

VN equation can be represented as compatibility condition in the form of Manakov's triad [12]:

$$[L_1, L_2] = BL_1, \quad B = 3\big(\kappa\partial_{\bar{z}}^{-1}u_{zz} + \overline{\kappa}\partial_z^{-1}u_{\bar{z}\bar{z}}\big) \tag{1.2}$$

of two linear auxiliary problems

$$L_1\psi = \big(\partial_{z\bar{z}}^2 + u\big)\psi = 0, \tag{1.3}$$

$$L_2\psi = \big(\partial_t + \kappa\partial_z^3 + \overline{\kappa}\partial_{\bar{z}}^3 + 3\kappa\big(\partial_{\bar{z}}^{-1}u_z\big)\partial_z + 3\overline{\kappa}\big(\partial_z^{-1}u_{\bar{z}}\big)\partial_{\bar{z}}\big)\psi = 0. \tag{1.4}$$

Several classes of exact solutions of VN equation (1.1) have been constructed in last three decades (1980 − 2010) via different methods [10, 11, 13–24], see also the books [3,4]. These solutions include finite-zone type solutions [11], rationally localized solutions [13–17, 20, 21] or lumps, solutions with functional parameters [10, 18, 23, 24], multi-line soliton solutions [22, 24] and so on. Underline that the first auxiliary linear

problem (1.3) is nothing but the 2D stationary Schrödinger equation so exact solutions of VN equation constructed via all IST approaches are also transparent potentials of this Schrödinger equation.

Recently in the paper [23] the class of exact solutions with functional parameters and constant asymptotic values $-\epsilon$ at infinity

$$u(z,\bar{z},t) = \tilde{u}(z,\bar{z},t) - \epsilon, \quad \tilde{u}(z,\bar{z},t)|_{|z|\to\infty} \to 0 \tag{1.5}$$

of VN equation (1.1) via $\overline{\partial}$-dressing method of Zakharov and Manakov [6–9] has been calculated and subclass of multi-line soliton solutions has been presented [23].

In another paper [24] (see also [22]) it was established that for some special solutions $u^{(1)}$ and $u^{(2)}$, i. e. for special linear (plane) solitons or for special plane wave type singular periodic solutions, with zero value $E = -2\epsilon = 0$ of energy level of corresponding 2D stationary Schrödinger equation, their sum $u^{(1)} + u^{(2)}$ is also exact solution of VN equation.

In present paper this result [22, 24] is generalized to the case of linear superpositions of arbitrary number of special line solitons (or special plane wave type periodic solutions) $u^{(n)}$, $n = 1,\ldots,N$ in such a way, that the sums of arbitrary subsets of these solutions

$$u = u^{(k_1)} + \ldots + u^{(k_m)}, \quad 1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant N \tag{1.6}$$

are also exact solutions of VN equation (1.1).

For convenience here some useful formulas of $\overline{\partial}$-dressing method for VN equation (1.1) [3, 4, 20–24] are presented. Central object of this method is the scalar wave function $\chi(\lambda; z, \bar{z}, t)$

$$\chi(\lambda; z,\bar{z},t) = e^{-F(\lambda;z,\bar{z},t)}\psi(z,\bar{z},t), \quad F(\lambda;z,\bar{z},t) = i\big[\lambda z - \frac{\epsilon}{\lambda}\bar{z} + \big(\kappa\lambda^3 - \overline{\kappa}\frac{\epsilon^3}{\lambda^3}\big)t\big] \tag{1.7}$$

which satisfies to corresponding $\overline{\partial}$-problem or equivalently to following singular integral equation:

$$\chi(\lambda) = 1 + \int\!\!\int_C \frac{d\lambda' \wedge d\overline{\lambda'}}{2\pi i(\lambda' - \lambda)} \int\!\!\int_C \chi(\mu,\overline{\mu})R(\mu,\overline{\mu};\lambda',\overline{\lambda'})d\mu \wedge d\overline{\mu}. \tag{1.8}$$

Here canonical normalization $\chi \to \chi_\infty = 1$ as $\lambda \to \infty$ of wave function is assumed and the kernel R is given by the formula [4, 22–24]

$$R(\mu,\overline{\mu};\lambda,\bar{\lambda};z,\bar{z},t) = R_0(\mu,\overline{\mu};\lambda,\bar{\lambda})e^{F(\mu;z,\bar{z},t)-F(\lambda;z,\bar{z},t)}. \tag{1.9}$$

Solutions $u(z,\bar{z},t)$ of VN equation are expressed via reconstruction formulas [4, 20–24]:

$$u = -\epsilon + i\epsilon\chi_{1z} = -\epsilon - i\chi_{-1\bar{z}} \tag{1.10}$$

through the coefficients χ_1 and χ_{-1} of Taylor's series

$$\chi = \chi_0 + \chi_1\lambda + \chi_2\lambda^2 + \ldots, \quad \chi = \chi_\infty + \frac{\chi_{-1}}{\lambda} + \frac{\chi_{-2}}{\lambda^2} + \ldots \tag{1.11}$$

expansions in the neighborhoods of points $\lambda = 0$ and $\lambda = \infty$ of complex plane $\mathbb{C}$. In constructing of exact solutions u of VN equation (1.1) two conditions must be satisfied [4, 20–24]: the condition of potentiality of operator L_1, or the absence in the first auxiliary linear problem (1.3) of the terms with first derivatives $u_1\partial_z$ and $u_2\partial_{\bar{z}}$, and the condition of reality $u = \bar{u}$ of solutions.

The potentiality condition on operator L_1 in (1.3), or equivalently in terms of wave function χ the condition $\chi_0 = 1$ [4, 20–24], imposes severe restrictions on the

kernel R_0 of $\overline{\partial}$-problem. The condition of reality of solutions $u = \bar{u}$ leads to another following restriction on the kernel R_0 [20–24]:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\epsilon^3}{|\mu|^2 |\lambda|^2 \overline{\mu}\overline{\lambda}} \overline{R_0\left(-\frac{\epsilon}{\overline{\lambda}}, -\frac{\epsilon}{\lambda}, -\frac{\epsilon}{\overline{\mu}} - \frac{\epsilon}{\mu}\right)}, \tag{1.12}$$

this restriction was obtained in the limit of "weak" fields. Both conditions were successfully applied in calculations of broad classes of exact solutions of VN equation (1.1) such as lumps [20, 21], solutions with functional parameters, multi-line solitons and plane wave type singular periodic solutions [22–24].

In the present note we do not use the limit of weak fields and impose the reality condition $u = \overline{u}$ directly to calculated complex solutions satisfying only to potentiality condition. This approach makes it possible to receive besides multi-line soliton solutions also plane wave type singular periodic solutions (this was shown at first in [22, 24]) and their's superpositions.

By the application of $\overline{\partial}$-dressing in the special limit of zero energy level we obtain in present paper new exact solutions, nonstationary and stationary, of VN equation in the forms of linear superpositions of special line solitons and also linear superpositions of special plane wave type singular periodic solutions. By construction these exact solutions represent also new exact transparent potentials of 2D stationary Schrödinger equation and can find an applications as model potentials for electrons in planar structures of modern electronics.

2. Nonlinear superpositions of complex solutions of VN equation

The choice of delta-functional kernel

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_n A_n \delta(\mu - \mathrm{M}_n)\delta(\lambda - \Lambda_n) \tag{2.1}$$

with complex constant coefficients A_n and complex discrete spectral parameters $\mathrm{M}_n \neq \Lambda_n$ leads to simple determinant formula [22, 24]

$$u = -\epsilon + \frac{\partial^2}{\partial z \partial \bar{z}} \ln \det A, \quad A_{lk} = \delta_{lk} + \frac{2iA_k}{\mathrm{M}_l - \Lambda_k} e^{F(\mathrm{M}_l) - F(\Lambda_k)} \tag{2.2}$$

for exact multi-line soliton and plane wave type singular periodic solutions of VN equation. The main problem in using this formula is satisfaction to reality and potentiality conditions.

It was shown in [22, 24] that the choice of kernel R_0 (2.1) in the form

$$R_0(\mu, \bar{\mu}; \lambda, \bar{\lambda}) = \pi \sum_{n=1}^{N} \Big[a_n \lambda_n \delta(\mu - \mu_n)\delta(\lambda - \lambda_n) + a_n \mu_n \delta(\mu + \lambda_n)\delta(\lambda + \mu_n)\Big] \tag{2.3}$$

of N paired terms with discrete spectral parameters (μ_n, λ_n) allows to satisfy the potentiality condition $\chi_0 = 1$. In the simplest cases $N = 1, 2$ one obtains from (2.1) – (2.3) the following expressions for $\det A$ [22, 24]:

$$N = 1 : \det A = \Big(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)}\Big)^2, \tag{2.4}$$

$$N = 2 : \det A = \Big(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)} + s_n e^{\Delta F(\mu_n, \lambda_n)} + w e^{\Delta F(\mu_1, \lambda_1) + \Delta F(\mu_n, \lambda_n)}\Big)^2. \tag{2.5}$$

Here, in (2.3) – (2.5) a_n, μ_n, λ_n $(n = 1, \ldots, N)$ are some complex constants; μ_n and λ_n also known as discrete spectral parameters which give spectral characterization for

corresponding exact solutions. The quantities s_n, w and $\Delta F(\mu_n, \lambda_n)$ are given by the formulas:

$$s_n := ia_n \frac{\mu_n + \lambda_n}{\mu_n - \lambda_n}; \quad \Delta F(\mu_n, \lambda_n) := F(\mu_n) - F(\lambda_n), \tag{2.6}$$

$$w := -s_1 s_n \cdot \frac{(\lambda_1 - \lambda_n)(\lambda_n + \mu_1)(\mu_1 - \mu_n)(\lambda_1 + \mu_n)}{(\lambda_1 + \lambda_n)(\lambda_n - \mu_1)(\mu_1 + \mu_n)(\lambda_1 - \mu_n)}. \tag{2.7}$$

The expression for $\det A$ in the case $N = 2$ (2.5) is generated by two pairs of terms in (2.3) with discrete spectral variables (μ_1, λ_1) and (μ_n, λ_n).

The formula for generally complex solution corresponding to one arbitrary pair $(\mu_m, \lambda_m), (m = 1, \ldots, N)$ of discrete spectral variables due to (2.1) – (2.4) and (2.6) has the form:

$$u^{(m)}(z, \bar{z}, t) = -\epsilon + \tilde{u}^{(m)}(z, \bar{z}, t) = -\epsilon - \epsilon \frac{2s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2}. \tag{2.8}$$

It is remarkable that for $w = s_1 s_n$ in (2.5) for case $N = 2$ of two pairs of terms in (2.3) with spectral variables (μ_1, λ_1) and $(\mu_n, \lambda_n), (n = 2, \ldots, N)$, i. e. for the choice

$$\frac{(\lambda_1 - \lambda_n)(\lambda_n + \mu_1)(\mu_1 - \mu_n)(\lambda_1 + \mu_n)}{(\lambda_1 + \lambda_n)(\lambda_n - \mu_1)(\mu_1 + \mu_n)(\lambda_1 - \mu_n)} = -1, \tag{2.9}$$

which is equivalent to relation

$$(\lambda_1 - \mu_1)(\lambda_n - \mu_n)(\lambda_1 \mu_1 + \lambda_n \mu_n) = 0, \quad n \neq 1, \tag{2.10}$$

the expression for $\det A$ (2.5) greatly simplifies

$$\det A = \left(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)}\right)^2 \left(1 + s_n e^{\Delta F(\mu_n, \lambda_n)}\right)^2. \tag{2.11}$$

The solutions $\mu_1 = \lambda_1$ and $\mu_n = \lambda_n$ of (2.10) correspond to lumps (rationally decreasing at infinity exact solutions u [20, 21] of VN equation) and in accordance with $M_n \neq \Lambda_n$ in (2.1) will not be considered here, so it is assumed below that $\mu_m \neq \lambda_m$ in (2.3) for all terms $m = 1, \ldots, N$ with discrete spectral variables μ_m, λ_m. Under this requirement the relations (2.9), (2.10) reduce to more simple ones:

$$\lambda_n \mu_n + \lambda_1 \mu_1 = 0, \quad n = 2, \ldots, N. \tag{2.12}$$

An application of general formulas (2.1) – (2.5) in the case $N = 2$ due to (2.9) or (2.12) leads to very simple expression for complex solution of VN equation

$$u(z, \bar{z}, t) = -\epsilon - 2\epsilon \sum_{m=1,n} \frac{s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2} \tag{2.13}$$

which is nonlinear superposition $u = \epsilon + u^{(1)} + u^{(n)}$ of two solutions $u^{(1)}$ and $u^{(n)}$ of the type (2.8) with corresponding pairs of spectral variables (μ_1, λ_1) and (μ_n, λ_n). Up to the constant ϵ the solution (2.13) is the sum of complex solutions $u^{(1)}$ and $u^{(n)}$.

It is easy to prove also that nonlinear superposition

$$u(z, \bar{z}, t) = -\epsilon + \tilde{u}^{(1)}(z, \bar{z}, t) + \sum_{m=2}^{N} \tilde{u}^{(m)}(z, \bar{z}) = -\epsilon - 2\epsilon \frac{s_1(\mu_1 - \lambda_1)^2}{\mu_1 \lambda_1} \frac{e^{\Delta F(\mu_1, \lambda_1)}}{(1 + s_1 e^{\Delta F(\mu_1, \lambda_1)})^2} -$$

$$-2\epsilon \sum_{m=2}^{N} \frac{s_m(\mu_m - \lambda_m)^2}{\mu_m \lambda_m} \frac{e^{\Delta F(\mu_m, \lambda_m)}}{(1 + s_m e^{\Delta F(\mu_m, \lambda_m)})^2} \tag{2.14}$$

of arbitrary number $N \geqslant 2$ of complex solutions, solution $u^{(1)}(z,\bar{z},t) = -\epsilon + \tilde{u}^{(1)}(z,\bar{z},t)$ and $N-1 \geqslant 1$ solutions $u^{(m)}(z,\bar{z}) = -\epsilon + \tilde{u}^{(m)}(z,\bar{z})$, $m = 2,\ldots,N$ of the type (2.8) is also exact solution of VN equation, when conditions (2.12) are fulfilled and parameters μ_1 and λ_1 are satisfied to additional restriction

$$\kappa\lambda_1^3 - \overline{\kappa}\frac{\epsilon^3}{\mu_1^3} = 0. \tag{2.15}$$

Due to conditions (2.12), (2.15) the phases $\Delta F(\mu_m,\lambda_m)$ (2.6) in (2.14) take the forms:

$$\varphi_1(z,\bar{z},t) := \Delta F(\mu_1,\lambda_1) = i\left[(\mu_1-\lambda_1)z - \left(\frac{\epsilon}{\mu_1} - \frac{\epsilon}{\lambda_1}\right)\bar{z} - 2\left(\kappa\lambda_1^3 - \overline{\kappa}\frac{\epsilon^3}{\lambda_1^3}\right)t\right], \tag{2.16}$$

$$\varphi_m(z,\bar{z}) := \Delta F(\mu_m,\lambda_m) = i\left[(\mu_m-\lambda_m)z - \left(\frac{\epsilon}{\mu_m} - \frac{\epsilon}{\lambda_m}\right)\bar{z}\right], \quad m = 2,\ldots,N. \tag{2.17}$$

These expressions (2.16) and (2.17) mean that the first complex solution $u^{(1)}(z,\bar{z},t) = -\epsilon + \tilde{u}^{(1)}(z,\bar{z},t)$ of superposition (2.14) propagates with nonzero velocity in the plane (x,y) but all other complex solutions $u^{(m)}(z,\bar{z}) = -\epsilon + \tilde{u}^{(m)}(z,\bar{z})$, $(m = 2,\ldots,N)$ of superposition (2.14) with $N \geqslant 2$ are fixed in the plane (x,y) stationary solutions of VN equation (1.1).

One can prove also that every subsum of arbitrary terms $1 \leqslant i < i+1 < \ldots < j-1 < j \leqslant N$ of sum (2.14)

$$u = -\epsilon + \sum_{n=i}^{j} \tilde{u}^{(n)} \tag{2.18}$$

under conditions (2.12) and (2.15) is exact solution of VN equation. Complex solutions of VN equation given by (2.18) due to (2.14) and (2.16), (2.17) can be divided on two classes: the class of nonstationary solutions with $i \geqslant 1$ and class of stationary solutions with $i \geqslant 2$.

3. Linear superpositions of line soliton solutions for Veselov-Novikov equation

For construction of real multi-line solitons via (2.2) besides potentiality condition satisfied by the kernel R_0 of the type (2.3) the reality condition $u = \overline{u}$ for solutions u must be fulfilled. This can be done choosing appropriately complex constants a_n and complex discrete spectral parameters (μ_n,λ_n) in (2.3) – (2.18) by several ways [22,24]. For example, by imposing reality condition $u = \bar{u}$ on complex solutions (2.8), (2.13), (2.14) and (2.18) with additional assumption of real phases $\Delta F(\mu_n,\lambda_n) = \overline{\Delta F(\mu_n,\lambda_n)}$ (2.6) we have calculated real multi-line soliton solutions.

It was shown in the papers [22,24] that to such real solutions u leads the following choice of parameters

$$a_n = -\overline{a}_n := ia_{n0}, \quad \mu_n = -\frac{\epsilon}{\overline{\lambda}_n}, \quad n = 1,\ldots,N \tag{3.1}$$

with real constants a_{n0}. Due to (3.1) and under additional assumption of positive values of real constants s_n given by (2.6)

$$s_n = a_{n0}\frac{\lambda_n+\mu_n}{\lambda_n-\mu_n} \stackrel{\text{def}}{=} e^{\phi_{0n}} > 0, \quad n = 1,\ldots,N \tag{3.2}$$

the solution (2.8) corresponding to one arbitrary pair (μ_n, λ_n), $(n = 1, \dots, N)$ of discrete spectral variables takes the form of real nonsingular one-line soliton solution:

$$u^{(n)}(x,y,t) = -\epsilon + \tilde{u}^{(n)}(x,y,t) = -\epsilon + \frac{|\lambda_n - \mu_n|^2}{2\cosh^2 \frac{\varphi_n(x,y,t)+\phi_{0n}}{2}} \tag{3.3}$$

where real phases $\varphi_n(x,y,t) := \Delta F(\mu_n, \lambda_n)$ (2.6) due to (3.1) have the form

$$\varphi_n(x,y,t) = 2|\lambda_n|\left(1 + \frac{\epsilon}{|\lambda_n|^2}\right)(\vec{N}_n\vec{r} - V_n t), \quad n = 1, \dots, N, \tag{3.4}$$

here $\vec{r} = (x, y)$, $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\varphi_n(x,y,t)$ and V_n are corresponding velocities of one-line solitons

$$\vec{N}_n = \left(\frac{\lambda_{nI}}{|\lambda_n|}, \frac{\lambda_{nR}}{|\lambda_n|}\right), \quad V_n = -\frac{1}{|\lambda_n|}\left(1 + \frac{\epsilon\left(\epsilon - |\lambda_n|^2\right)}{|\lambda_n|^4}\right)\mathrm{Im}(\kappa\lambda_n^3), \quad n = 1, \dots, N. \tag{3.5}$$

For the cases of nonlinear superpositions (2.13) $(N = 2)$, (2.14) and (2.18) $(N \geqslant 2)$ of exact solutions of the type (3.3) the conditions (2.12) for discrete spectral parameters (μ_n, λ_n), $(n > 1)$ due to (3.1) lead to following parametrization of (μ_n, λ_n)

$$\mu_n = i\tau_n^{-1}\mu_1, \quad \lambda_n = i\tau_n\lambda_1, \quad n = 2, \dots, N \tag{3.6}$$

with arbitrary real constants τ_n. Nonsingular two-line soliton solution characterized by two pairs of discrete spectral variables (μ_1, λ_1) and (μ_2, λ_2) due to (2.13), (3.1), (3.2) and (3.6) takes the form

$$u(x,y,t) = -\epsilon + \sum_{n=1}^{2}\tilde{u}^{(n)}(x,y,t) = -\epsilon + \sum_{n=1}^{2}\frac{|\lambda_n - \mu_n|^2}{2\cosh^2 \frac{\varphi_n(x,y,t)+\phi_{0n}}{2}}, \tag{3.7}$$

where $u^{(n)}(x,y,t) = -\epsilon + \tilde{u}^{(n)}(x,y,t)$, $(n = 1, 2)$ are one-line soliton solutions of the type (3.3), the phases $\varphi_n(x,y,t), (n = 1, 2)$ under conditions (3.6) are given by (3.4). Due to expressions for vectors of normals (3.5) and parametrization (3.6) it is evident that solitons $u^{(1)}$ and $u^{(2)}$ of superposition (3.7) move in the plane (x, y) perpendicularly to each other.

One of two one-line solitons $u^{(1)}$ or $u^{(2)}$ (not both) of superposition (3.7) due to (3.5) and (3.6) can be "stopped", i. e. by special choice of spectral parameter λ_1 one can take ones of velocities $V_1 = 0$ or $V_2 = 0$ equal to zero. For example one can choose $V_2 = 0$, this achieves by the use of (3.5) and (3.6) for λ_1 satisfying to condition

$$\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1^3} = 0. \tag{3.8}$$

It was shown in the papers [22, 24] that the limiting procedure for calculation of exact solutions u of VN equation with zero asymptotic values at infinity $u|_{|z|^2\to\infty} = -\epsilon \to 0$ ($\overline{\partial}$-dressing on zero energy level) can be defined by the following way

$$\epsilon \to 0, \quad \mu_n \to 0, \quad \frac{\epsilon}{\mu_n} \to -\overline{\lambda}_n \neq 0, \quad n = 1, \dots, N. \tag{3.9}$$

It is assumed that under procedure (3.9) the relations $\lambda_n = i\tau_n\lambda_1$ from (3.6) remain to be valid.

In the limit (3.9) two-line soliton solution (3.7) converts to linear superposition

$$u(x,y,t) = u^{(1)}_{\epsilon=0} + u^{(2)}_{\epsilon=0} = \sum_{n=1}^{2}\frac{|\lambda_n|^2}{2\cosh^2 \frac{\tilde{\varphi}_n(x,y,t)+\phi_{0n}}{2}} \tag{3.10}$$

of two one-line solitons $u^{(1)}_{\epsilon=0}$ and $u^{(2)}_{\epsilon=0}$

$$u^{(n)}_{\epsilon=0}(x,y,t) = \frac{|\lambda_n|^2}{2\cosh^2\frac{\tilde{\varphi}_n(x,y,t)+\phi_{0n}}{2}}, \quad n=1,2 \tag{3.11}$$

with phases $\tilde{\varphi}_n(x,y,t)$ given due to (3.4) and (3.9) by formulas

$$\tilde{\varphi}_n(x,y,t) = 2|\lambda_n|(\vec{N}_n\vec{r} - V_n t), \quad \vec{N}_n = \left(\frac{\lambda_{nI}}{|\lambda_n|}, \frac{\lambda_{nR}}{|\lambda_n|}\right), \quad n=1,2. \tag{3.12}$$

Here $\vec{r} = (x,y)$, $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x,y,t)$; V_n are corresponding velocities of one-line solitons

$$V_1 = -\frac{\mathrm{Im}(\kappa\lambda_1^3)}{|\lambda_1|}, \quad V_2 = -\frac{\mathrm{Im}(\kappa\lambda_2^3)}{|\lambda_2|} = \frac{\tau_2^3\mathrm{Re}(\kappa\lambda_1^3)}{|\tau_2||\lambda_1|} \tag{3.13}$$

derived by the use of (3.5), (3.6) and (3.9). By special choice of spectral parameter λ_1 one of two one-line solitons $u^{(1)}_{\epsilon=0}$ or $u^{(2)}_{\epsilon=0}$ (not both) in linear superposition (3.10) can be "stopped".

The solution in the form of another nonlinear superposition of $N \geqslant 2$ one-line solitons (3.3) is given by (2.14) with parameters a_n, (μ_n, λ_n) and s_n satisfying to (3.1), (3.2), the conditions (2.12) or (3.6) and (2.15) also must be fulfilled. The condition (2.15) due to (3.1) leads to relation

$$\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1^3} = 0, \tag{3.14}$$

so the solution (2.14) takes the following real form

$$u(x,y,t) = -\epsilon + \frac{|\lambda_1-\mu_1|^2}{2\cosh^2\frac{\varphi_1(x,y,t)+\phi_{01}}{2}} + \sum_{n=2}^{N}\frac{|\lambda_n-\mu_n|^2}{2\cosh^2\frac{\varphi_n(x,y)+\phi_{0n}}{2}} \tag{3.15}$$

with phases φ_n (2.16), (2.17) are given due to (3.1), (3.2) and (3.6), (3.14) by expressions

$$\varphi_1(x,y,t) = 2|\lambda_1|\left(1+\frac{\epsilon}{|\lambda_1|^2}\right)(\vec{N}_1\vec{r} - V_1 t),\ \varphi_n(x,y) = 2|\lambda_1|\left(\tau_n + \frac{\epsilon}{\tau_n|\lambda_1|^2}\right)(\vec{N}_2\vec{r}). \tag{3.16}$$

In formulas (3.16) $n = 2,\ldots,N$, $\vec{r} = (x,y)$; the unit vectors of normals $\vec{N}_1$, $\vec{N}_2$ and velocity V_1 are given by following formulas

$$\vec{N}_1 = \left(\frac{\lambda_{1I}}{|\lambda_1|}, \frac{\lambda_{1R}}{|\lambda_1|}\right), \quad \vec{N}_2 = \left(\frac{\lambda_{1R}}{|\lambda_1|}, -\frac{\lambda_{1I}}{|\lambda_1|}\right), \quad V_1 = \frac{i\kappa\lambda_1^3}{|\lambda_1|}\left(1+\frac{\epsilon\left(\epsilon-|\lambda_1|^2\right)}{|\lambda_1|^4}\right). \tag{3.17}$$

One-line soliton $u^{(1)}(x,y,t) = -\epsilon + \tilde{u}^{(1)}(x,y,t)$ of nonlinear superposition (3.15) due to (3.16) and (3.17) moves in the plane (x,y) perpendicularly to others stationary solitons $u^{(n)}(x,y) = -\epsilon + \tilde{u}^{(n)}(x,y)$, $(n = 2,\ldots,N)$ of this superposition with parallel lines of constant values of phases $\varphi_n(x,y)$. Evidently particular case of (3.7) with $V_2 = 0$ due to (3.8) and (3.14) coincides with two-line soliton (for $N = 2$) nonlinear superposition (3.15).

In the limit $\epsilon \to 0$ following to the rules (3.9), under requirement that relations $\lambda_n = i\tau_n\lambda_1$ from (3.6) and (3.14) remain to be valid, we obtain from (3.15) linear superposition of N one-line solitons

$$u = u^{(1)}_{\epsilon=0}(x,y,t) + \sum_{n=2}^{N} u^{(n)}_{\epsilon=0}(x,y) = \frac{|\lambda_1|^2}{2\cosh^2\frac{\tilde{\varphi}_1(x,y,t)+\phi_{01}}{2}} + \sum_{n=2}^{N}\frac{|\lambda_n|^2}{2\cosh^2\frac{\tilde{\varphi}_n(x,y)+\phi_{0n}}{2}}, \tag{3.18}$$

here the phases $\tilde{\varphi}_n$ given by expressions

$$\tilde{\varphi}_1(x,y,t) = 2|\lambda_1|\left(\vec{N}_1\vec{r} - \frac{i\kappa\lambda_1^3}{|\lambda_1|}t\right), \quad \tilde{\varphi}_n(x,y) = 2\tau_n|\lambda_1|(\vec{N}_2\vec{r}), \quad n = 2,\ldots,N \quad (3.19)$$

are obtained from phases φ_n (3.16) in limit $\epsilon \to 0$ (3.9). The first one-line soliton $u^{(1)}_{\epsilon=0}(x,y,t)$ from linear superposition (3.18) moves with velocity $V_1 = \frac{i\kappa\lambda_1^3}{|\lambda_1|}$ in the plane (x,y) perpendicularly to other $(N-1)$ stationary one-line solitons $u^{(n)}_{\epsilon=0}(x,y)$.

One can prove also that the subsums of arbitrary numbers of solitons $u^{(n)}_{\epsilon=0}$, $(n = 1,\ldots,N)$ from (3.18) are also solutions of VN equation. The set of such solutions can be divided in two subsets: subset of nonstationary linear superpositions (with the first moving line soliton $u^{(1)}_{\epsilon=0}(x,y,t)$ in the sum) of line solitons and subset of stationary linear superpositions (without moving line soliton $u^{(1)}_{\epsilon=0}(x,y,t)$ in the sum) of stationary line solitons.

4. Linear superpositions of plane wave type periodic solutions for Veselov-Novikov equation

Multi-line soliton solutions are calculated in preceding section by imposing reality condition $u = \bar{u}$ on complex solutions (2.8), (2.13), (2.14) and (2.18) with additional assumption of real phases $\Delta F(\mu_n,\lambda_n) = \overline{\Delta F(\mu_n,\lambda_n)}$ (2.6). In contrast the application of reality condition $u = \bar{u}$ with assumption of pure imaginary phases $\Delta F(\mu_n,\lambda_n) = -\overline{\Delta F(\mu_n,\lambda_n)}$ (2.6) leads to plane wave type periodic solutions and their's superpositions.

Plane wave type solutions can be obtained by this way for example by the following choice of parameters $a_n, (\mu_n,\lambda_n)$ and s_n in (2.3) – (2.18) [22, 24]:

$$\mu_n = \frac{\epsilon}{\overline{\lambda}_n},\ a_n = \left|\frac{\lambda_n - \mu_n}{\lambda_n + \mu_n}\right| e^{i\arg a_n},\ s_n = -ie^{i\arg a_n}\mathrm{sign}\left(\frac{\lambda_n - \mu_n}{\lambda_n + \mu_n}\right),\ n = 1,\ldots,N. (4.1)$$

Simple plane wave type periodic solution corresponding to one arbitrary pair of spectral variables (μ_n,λ_n), $(n = 1,...,N)$, due to (2.8) and (4.1) takes the form

$$u^{(n)}(x,y,t) = -\epsilon + \tilde{u}^{(n)}(x,y,t) = -\epsilon - \frac{|\lambda_n - \mu_n|^2}{2\cos^2\left(\frac{\varphi_n(x,y,t)+\arg a_n}{2} \mp \frac{\pi}{4}\right)}, \quad (4.2)$$

where sign "$-$" correspond to case of $|\lambda_n| > |\mu_n|$ and sign "$+$" – case of $|\lambda_n| < |\mu_n|$. The real phases $\varphi_n(x,y,t) := -i\Delta F(\mu_n,\lambda_n)$ in (4.2) due to (2.6) and (4.1) are given by expressions

$$\varphi_n(x,y,t) = 2|\lambda_n|\left(\frac{\epsilon}{|\lambda_n|^2} - 1\right)(\vec{N}_n\vec{r} - V_n t), \quad n = 1,\ldots,N. \quad (4.3)$$

In (4.2), (4.3) $\vec{r} = (x,y)$; $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\varphi_n(x,y,t)$ and velocities V_n of periodic solutions are given by expressions:

$$\vec{N}_n = \left(\frac{\lambda_{nR}}{|\lambda_n|}, -\frac{\lambda_{nI}}{|\lambda_n|}\right),\ V_n = -\frac{1}{|\lambda_n|}\left(1 + \frac{\epsilon\left(\epsilon + |\lambda_n|^2\right)}{|\lambda_n|^4}\right)\mathrm{Re}(\kappa\lambda_n^3),\ n = 1,\ldots,N. \quad (4.4)$$

Using general formulas (2.13) and (2.14) we also construct nonlinear superpositions of simple wave type periodic solutions of the type (4.2). The conditions (2.12) for discrete spectral parameters (μ_n,λ_n), $(n > 1)$ in nonlinear superpositions

(2.13) and (2.14) due to (4.1) in considered case lead to following parametrization of (μ_n, λ_n)

$$\lambda_n = i\tau_n\lambda_1, \quad \mu_n = i\tau_n^{-1}\mu_1, \quad n = 2, \dots, N \tag{4.5}$$

with arbitrary real constants τ_n. Nonlinear superposition (2.13) of two simple plane wave type periodic solutions of the type (4.2) due to (4.1) and (4.5) takes the form

$$u(x,y,t) = -\epsilon + \sum_{n=1}^{2} \tilde{u}^{(n)}(x,y,t) = -\epsilon - \sum_{n=1}^{2} \frac{|\lambda_n - \mu_n|^2}{2\cos^2\left(\frac{\varphi_n(x,y,t)+\arg a_n}{2} \mp \frac{\pi}{4}\right)}, \tag{4.6}$$

where sign "$-$" correspond to case of $|\lambda_n| > |\mu_n|$ and sign "$+$" – case of $|\lambda_n| < |\mu_n|$. The phases $\varphi_n(x,y,t)$ in solution (4.6) are given by (4.3) with parametrization (4.5). Due to expressions for vectors of normals (4.4) and to parametrization (4.5) it is evident that lines of constant values of phases $\varphi_n(x,y,t), (n = 1,2)$ for solutions $u^{(1)} = -\epsilon + \tilde{u}^{(1)}$ and $u^{(2)} = -\epsilon + \tilde{u}^{(2)}$ of superposition (4.6) move perpendicularly to each other.

One of two simple plane wave type periodic solutions $u^{(1)}$ or $u^{(2)}$ (not both) of superposition (4.6) due to (4.4) and (4.5) can be made stationary, i. e. by special choice of spectral parameter λ_1 one can take ones of velocities $V_1 = 0$ or $V_2 = 0$ equal to zero. For example one can choose $V_2 = 0$, this achieves due to (4.4) and (4.5) for λ_1 satisfying to condition

$$\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3 = 0. \tag{4.7}$$

It was shown in the papers [22, 24] that the limiting procedure of calculation of exact solutions u of VN equation with zero values of parameter $\epsilon = 0$ ($\overline{\partial}$-dressing on zero energy level) can be defined by the following way

$$\epsilon \to 0, \quad \mu_n \to 0, \quad \frac{\epsilon}{\mu_n} \to \overline{\lambda}_n \neq 0, \quad n = 1, \dots, N. \tag{4.8}$$

Such procedure is applicable also in considered case of plane wave type solutions and their's superpositions. It is assumed that under procedure (4.8) the relations $\lambda_n = i\tau_n\lambda_1$ from (4.5) remain to be valid.

In the limit (4.8) nonlinear superposition (4.6) of two plane wave type periodic solutions converts to linear superposition

$$u(x,y,t) = u^{(1)}_{\epsilon=0} + u^{(2)}_{\epsilon=0} = -\sum_{n=1}^{2} \frac{|\lambda_n|^2}{2\cos^2\left(\frac{\tilde{\varphi}_n(x,y,t)+\arg a_n}{2} - \frac{\pi}{4}\right)} \tag{4.9}$$

of two periodic solitons $u^{(1)}_{\epsilon=0}$ and $u^{(2)}_{\epsilon=0}$

$$u^{(n)}_{\epsilon=0}(x,y,t) = -\frac{|\lambda_n|^2}{2\cos^2\left(\frac{\tilde{\varphi}_n(x,y,t)+\arg a_n}{2} - \frac{\pi}{4}\right)}, \quad n = 1,2, \tag{4.10}$$

the phases $\tilde{\varphi}_n(x,y,t)$ in (4.9), (4.10) are given due to (4.3) and (4.8) by formulas

$$\tilde{\varphi}_n(x,y,t) = -2|\lambda_n|(\vec{N}_n\vec{r} - V_n t), \quad \vec{N}_n = \left(\frac{\lambda_{nR}}{|\lambda_n|}, -\frac{\lambda_{nI}}{|\lambda_n|}\right), \quad n = 1,2, \tag{4.11}$$

here $\vec{r} = (x,y)$; $\vec{N}_n$ are unit vectors of normals to lines of constant values of phases $\tilde{\varphi}_n(x,y,t)$. The corresponding velocities V_n of simple plane wave type periodic solutions (4.10)

$$V_1 = -\frac{\mathrm{Re}(\kappa\lambda_1^3)}{|\lambda_1|}, \quad V_2 = -\frac{\mathrm{Re}(\kappa\lambda_2^3)}{|\lambda_2|} = \frac{\tau_2^3\mathrm{Im}(\kappa\lambda_1^3)}{|\tau_2||\lambda_1|} \tag{4.12}$$

are derived by the use of (4.4), (4.5) and (4.8). By special choice of spectral parameter λ_1 one of two of these periodic solutions, $u^{(1)}_{\epsilon=0}$ or $u^{(2)}_{\epsilon=0}$ (not both), in linear superposition (4.9) can be made stationary.

The solution in the form of another nonlinear superposition of $N \geqslant 2$ simple plane wave type periodic solutions (4.2) is given by (2.14) with parameters a_n, (μ_n, λ_n) and s_n satisfying to (4.1), the conditions (2.12) or (4.5) and (2.15) also must be fulfilled. The condition (2.15) due to (4.1) transforms into following

$$\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1^3} = 0, \tag{4.13}$$

so the solution (2.14) takes the form

$$u(x,y,t) = -\epsilon - \frac{|\lambda_1 - \mu_1|^2}{2\cos^2\left(\frac{\varphi_1(x,y,t)+\arg a_1}{2} \mp \frac{\pi}{4}\right)} - \sum_{n=2}^{N} \frac{|\lambda_n - \mu_n|^2}{2\cos^2\left(\frac{\varphi_n(x,y)+\arg a_n}{2} \mp \frac{\pi}{4}\right)}, \tag{4.14}$$

where sign "$-$" correspond to case of $|\lambda_n| > |\mu_n|$ and sign "$+$" – case of $|\lambda_n| < |\mu_n|$, $n = 1, \ldots, N$. The phases φ_n (2.16), (2.17) in superposition (4.14) due to (4.1) and (4.5), (4.13) are given by expressions

$$\varphi_1(x,y,t) = 2|\lambda_1|\left(\frac{\epsilon}{|\lambda_1|^2} - 1\right)(\vec{N}_1\vec{r} - V_1 t), \varphi_n(x,y) = 2|\lambda_1|\left(\frac{\epsilon}{\tau_n|\lambda_1|^2} - \tau_n\right)(\vec{N}_2\vec{r}), \tag{4.15}$$

where $n = 2, \ldots, N$, $\vec{r} = (x,y)$; unit vectors $\vec{N}_n$ and velocity V_1 are given by following formulas

$$\vec{N}_1 = \left(\frac{\lambda_{1R}}{|\lambda_1|}, -\frac{\lambda_{1I}}{|\lambda_1|}\right),\ \vec{N}_2 = \left(-\frac{\lambda_{1I}}{|\lambda_1|}, -\frac{\lambda_{1R}}{|\lambda_1|}\right),\ V_1 = \frac{\kappa\lambda_1^3}{|\lambda_1|}\left(1 + \frac{\epsilon\left(\epsilon + |\lambda_1|^2\right)}{|\lambda_1|^4}\right). \tag{4.16}$$

The lines of constant values of phase $\varphi_1(x,y,t)$ of simple periodic solution $u^{(1)}(x,y,t) = -\epsilon + \tilde{u}^{(1)}(x,y,t)$ of superposition (4.14) move in plane (x,y) perpendicularly to parallel lines of constant phases $\varphi_n(x,y)$ of others stationary periodic solutions $u^{(n)}(x,y,t) = -\epsilon + \tilde{u}^{(n)}(x,y,t)$, $(n = 2, \ldots, N)$ of this superposition. Evidently particular case of (4.6) with $V_2 = 0$ coincides due to (4.7) and (4.13) with the case $N = 2$ of linear superposition (4.14).

In the limit $\epsilon \to 0$ ($\overline{\partial}$-dressing on zero energy level) following to the rules (4.8), with assumption that relations $\lambda_n = i\tau_n\lambda_1$ from (4.5) and (4.13) remain to be valid, we obtain from (4.14) linear superposition of N simple plane wave type periodic solutions

$$u = u^{(1)}_{\epsilon=0}(x,y,t) + \sum_{n=2}^{N} u^{(n)}_{\epsilon=0}(x,y) =$$

$$= \frac{|\lambda_1|^2}{2\cos^2\left(\frac{\tilde{\varphi}_1(x,y,t)+\arg a_1}{2} - \frac{\pi}{4}\right)} + \sum_{n=2}^{N} \frac{|\lambda_n|^2}{2\cos^2\left(\frac{\tilde{\varphi}_n(x,y)+\arg a_n}{2} - \frac{\pi}{4}\right)}, \tag{4.17}$$

here phases $\tilde{\varphi}_n$ are obtained from phases φ_n (4.15) by setting $\epsilon = 0$ in accordance with (4.8), these phases have the forms:

$$\tilde{\varphi}_1(x,y,t) = -2|\lambda_1|\left(\vec{N}_1\vec{r} - \frac{\kappa\lambda_1^3}{|\lambda_1|}t\right),\ \tilde{\varphi}_n(x,y) = -2\tau_n|\lambda_1|\left(\vec{N}_2\vec{r}\right),\ n = 2, \ldots, N. \tag{4.18}$$

The lines of constant values of phase $\varphi_1(x,y,t)$ of the first periodic solution $u^{(1)}_{\epsilon=0}$ from linear superposition (4.17) move with velocity $V_1 = \frac{\kappa\lambda_1^3}{|\lambda_1|}$ perpendicularly to lines of constant phases $\varphi_n(x,y)$ of $(N-1)$ other stationary periodic solutions $u^{(n)}_{\epsilon=0}, (n = 2, \ldots, N)$ of this superposition.

One can prove also that the subsums of arbitrary numbers of solutions $u^{(n)}_{\epsilon=0}$, $(n = 1, \ldots, N)$ from (4.17) are also solutions of VN equation. So the set of constructed in present section solutions can be divided in two subsets: subset of nonstationary linear superpositions (with the first moving line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ in the sum) and subset of stationary linear superpositions (without moving line soliton $u^{(1)}_{\epsilon=0}(x, y, t)$ in the sum).

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